A robust 9-point ILU smoother for anisotropic problems

K. Johannsen, Heidelberg, Germany

Abstract Discrete systems arising from elliptic PDEs can be solved efficiently using multigrid methods. In many cases of practical interest the resulting linear equations exhibit strong anisotropies. It is well-known that standard multigrid methods fail to work for this type of problems. Various ILU methods have been proposed and investigated to overcome these difficulties. To be applied successfully, they usually require a modification of the ILU iteration [11, 13]. Only in the particular case of a 7-point decomposition for a 5-point discretization no modification is needed [8]. We give a new proof for this situation, showing in which way the smoothing property is related to the size of the restmatrix. The method is shown to carry over to 9-point finite element discretizations. Numerical experiments document the excellent smoothing properties.

AMS Subject Classification: 65F10, 65F50, 65N22, 65N30, 65N55.

Key words: singular perturbed problems, anisotropic problems, ILU factorization, robust multigrid method, sparse matrices, Fourier analysis

1 Introduction

Multigrid schemes are among the fastest methods for solving large systems of discrete equations arising from the discretization of PDEs. A frequently occurring problem in many practical applications is strongly anisotropic diffusion caused by the underlying model problem and/or by a discretization using anisotropic grids. It poses severe difficulties to multigrid methods due to a deterioration of the approximation in comparison to the algebraic properties. To overcome these problems, different strategies have been proposed. One method is based on semi-coarsening techniques, which improve the approximation [1, 2]. We shall not follow this direction here. A second strategy aims to improve the smoothing iteration. This idea, considered already in [2, 4, 12], has been rigorously defined in [13]. Essentially, the methods developed in this context are based on the idea of line-block iterations. A successful application is therefore limited to two-dimensional problems of anisotropic diffusion or three-dimensional problems with one strongly coupled spatial direction. This includes many problems from environmental sciences, where models often extend over large areas in horizontal direction only. In view on this type of problems, where the mathematical models often consist of systems of PDEs, methods applicable to systems are highly desirable.

Prominent schemes of the line smoother type are variants of the ILU iteration. They have been investigated thoroughly in a number of articles [6–11, 13]. Applied to the discretization of the anisotropic Laplace equation, the iteration usually has to be modified to fulfill a robust smoothing property [11, 13]. Only in the case of 7-point ILU smoothing for a 5-point discretization with a special ordering of the unknowns the unmodified ILU iteration can be applied as a smoother [8]. This case is of special interest and is investigated further in this paper. We give a proof of the smoothing property for the 5-point discretization using a 9-point ILU-decomposition stencil. The method coincides with the method investigated in [8]. The new proof relates the smoothing property to the size of the restmatrix of the iteration and allows the understanding of the robust smoothing property in these terms. Numerical experiments show that the method carries over to 9-point finite element discretizations. This is of special interest, since these methods are frequently used in numerical simulations. An extension of this method to discretizations resulting from systems of PDEs is easily possible.

The remainder of this paper is organized as follows. In section 2 we introduce the notation and define the model problem to be investigated. In section 3 we introduce the variants of the ILU iteration to be investigated. They are analyzed in section 4 using Fourier analysis. Numerical results are discussed in section 5, confirming the theoretical findings. Concluding remarks are given in section 6.

2 The model problem

Let $\Omega =]0,1[^2$, $\epsilon \in]0,1]$, $f \in \mathcal{L}_2(\Omega)$ and $u \in \mathcal{H}_0^1(\Omega)$ the weak solution of

$$
\begin{aligned}
-\epsilon\partial_{xx}u - \partial_{yy}u &= f, \quad \mathbf{x} \in \Omega, && \text{(1a)}\\
u &= 0, \quad \mathbf{x} \in \partial\Omega. && \text{(1b)}
\end{aligned}
$$

The solution is fully regular, i.e. $u \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$. Let $\mathcal{T}$ be a uniformly refined hierarchy of admissible, quasi-uniform triangulations and $\mathbb{T}_l \in \mathcal{T}$, $l \geq 0$. Let $V_l \subset \mathcal{H}_0^1(\Omega)$ denote the space spanned by conforming $\mathcal{P}_1$ Ansatz functions corresponding to $\mathbb{T}_l$. For the finite element solution $u_l(\epsilon)$ of (1) on grid $\mathbb{T}_l$ holds

$$
\|u(\epsilon) - u_l(\epsilon)\|_{\mathcal{L}_2(\Omega)} \leq C\epsilon^{\alpha} h_l^2 \|f\|_{\mathcal{L}_2(\Omega)}, \tag{2}
$$

where h_l denotes a characteristic length scale of $\mathbb{T}_l$, C is a generic constant independent of h_l, ϵ and f. The constant α depends on the triangulation. Standard arguments lead to $\alpha = 4$ [2]. For special grids, $\alpha = 1$ or $\alpha = 2$ may be shown [10].

To construct an ϵ-uniform convergent multigrid method using uniform grid refinement, the deterioration of the approximation with respect to ϵ has to be balanced by the behavior of the smoothing iteration. Let A_l denote the symmetric finite element stiffness matrix of problem (1) on grid level l and let the smoothing iteration be given by

$$
S_l := \mathbb{I}_l - W_l^{-1} A_l.
$$

$\mathbb{I}_l$ denotes the identity matrix and W_l a symmetric approximation of A_l. Let $R_l := W_l - A_l$, $\gamma > 0.5$ and

$$
\|R_l\|_2 \leq C\epsilon^{\alpha}\|A_l\|_2, \qquad W_l \geq \gamma A_l, \tag{3}
$$

then ϵ-uniform multigrid convergence of the $V(\nu,\nu)$-cycle for $\nu \geq 1$ follows from standard arguments, see [5].

For general grid hierarchies $\mathcal{T}$, i.e. $\alpha > 1$, a smoother of optimal complexity fulfilling (3) is not available. If the grid hierarchy matches certain conditions, a construction on the base of the ILU iterations is possible. We give the definition of an appropriate grid. Let $\mathbf{x}_1 = (0,0)$, $\mathbf{x}_2 = (1,0)$, $\mathbf{x}_3 = (0,1)$ and $\mathbf{x}_4 = (1,1)$ the corners of Ω and

$$
\begin{aligned}
\mathbb{T}_0^t &:= \{\text{triangle}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4),\ \text{triangle}(\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4)\}, && \text{(4a)}\\
\mathbb{T}_l^t &:= \text{the uniform refinement of } \mathbb{T}_{l-1}^t,\ l > 0. && \text{(4b)}
\end{aligned}
$$

The stiffness matrix A_l resulting from the discretization has a constant matrix stencil. Using the usual stencil notation, it can be written as

$$A_l = \begin{bmatrix} & -1 & 0 \\ -\epsilon & 2+2\epsilon & -\epsilon \\ 0 & -1 & \end{bmatrix}, \tag{5}$$

where the corresponding entries for boundary degrees of freedom (DOF) have to be omitted. The discretization fulfills the approximation property (2) with $\alpha = 1$ [10]. In the limiting case $\epsilon \to 0$ the problem decomposes into a set on one-dimensional problems with three point stencil.

For convenience of notation, the subscripts indicating the grid level will be omitted, whenever possible.

3 Variants of the ILU iteration

Let $A \in \mathbb{R}^{n\times n}$ and $I := \{1, \ldots, n\}$. A set $E \subset I \times I$ with $(i,i) \in E, \forall i \in I$ and $(i,j) \in E \Rightarrow (j,i) \in E$ is called a pattern. The ILU decomposition of A with respect to the pattern E is given by the multiplicative splitting

$$A = LU - R, \tag{6}$$

with matrices L and U uniquely determined by

$$(LU)_{i,j} = A_{i,j}, \quad \forall (i,j) \in E, \tag{7a}$$

$$diag(L) = \mathbb{I}, \quad (L)_{i,j} = (U)_{i,j} = 0, \quad \forall (i,j) \notin E \tag{7b}$$

$$L \text{ is lower triangular, } U \text{ is upper triangular matrix.} \tag{7c}$$

The decomposition can be calculated by an incomplete Gauss elimination and gives rise to the ILU iteration with iteration matrix

$$S := \mathbb{I} - (LU)^{-1}A.$$

Note, that LU is symmetric. To specify the ILU iteration completely, we need to specify the pattern and the ordering of the unknowns. We make use of the following two patterns

$$E_5 = \begin{bmatrix} & \star & \\ \star & \star & \star \\ & \star & \end{bmatrix}, \qquad E_9 = \begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}, \tag{8}$$

where a star denotes a possibly non-vanishing matrix entry. For boundary DOF the corresponding entries have to be omitted. Next, we specify two lexicographical orderings. In the first, we order the unknowns first top down and then left to right

$$x(n) < x(m) \quad \Rightarrow \quad n \prec m, \tag{9a}$$
$$x(n) = x(m) \wedge y(n) > y(m) \quad \Rightarrow \quad n \prec m, \tag{9b}$$

where n, m denote DOF, $x(n)$ the x-coordinate of n, $y(n)$ the y-coordinate of n and $n \prec m$, that n is enumerated before m. The second realizes an ordering first from left to right and then top down

$$y(n) > y(m) \quad \Rightarrow \quad n \prec m, \tag{10a}$$
$$y(n) = y(m) \wedge x(n) < x(m) \quad \Rightarrow \quad n \prec m. \tag{10b}$$

Now we can give the definitions of the ILU iterations used throughout the paper. Let the decomposition (6) be defined using the pattern E_5 from (8) and the ordering (9). We denote the resulting ILU iteration by the subscript 0 and abbreviate this definition by

$$S_0 \text{ is defined by pattern } E_5 \text{ and ordering (9)}$$

and analogously

$$S_1 \text{ is defined by pattern } E_9 \text{ and ordering (9)},$$
$$S_2 \text{ is defined by pattern } E_9 \text{ and ordering (10)}.$$

The corresponding lower-, upper- and rest-matrices are denoted by L_i, U_i and R_i, $i = 1, 2, 3$, resp.

In the context of discretization (5), iteration S_0 has been investigated in e.g. [13] and will be used here for comparison only; iteration S_1 and S_2 has been investigated in e.g. [8].

4 Fourier analysis

In this section we analyze the ILU iterations defined in the previous section by means of Fourier analysis. To this end, we have to restrict ourself to matrices with constant matrix stencils on infinite or cyclic domains. In this case the matrices commute pairwise and can be investigated by means of their spectrum. Let $h > 0$ and

$$g_h := \{(x_1, x_2) | x_1 = j_1 h, x_2 = j_2 h, j_1, j_2 \in \mathbb{Z}\}$$

an infinite grid. Let A be given by (5) and $k_1, k_2 \in]-\pi, \pi]$, then A is diagonal in the basis

$$\varphi_{k_1 k_2}(x, y) = \exp\left(i(k_1 x_1 + k_2 x_2)/h\right),$$

with the eigenvalues

$$A = \epsilon(2 - \cos(k_1)) + (2 - \cos(k_2)),$$

where we have identified the matrix with its eigenvalues. This representation is valid for infinite domains. In the case of finite domain the grid is a subset of g_h. For cyclic boundary conditions the admissible frequencies are a subset of $]-\pi, \pi]^2$. Let

$$\tilde{g}_h := \{(x_1, x_2) | (x_1, x_2) \in g_h \wedge x_1, x_2 \geq 0\}$$

the double semi-infinite grid. The ILU iteration cannot be defined on g_h directly. As mentioned above, the coefficients of L and U can be calculated from the recursion defined by the Gaussian elimination. This recursion can be applied to grid $\tilde{g}_h$. Therefore, for every ILU decomposition (6) specified in the previous section, we are able to define

$$\tilde{L}_{i,j} := \lim_{k,l \to \infty} L_{i+k,j+l}, \qquad \tilde{U}_{i,j} := \lim_{k,l \to \infty} U_{i+k,j+l}. \tag{11}$$

We claim, that the limits exist and define the constant matrix stencil of the ILU iteration on grid g_h by (11). In this way, we define $\tilde{L}_i$, $\tilde{U}_i$, $\tilde{R}_i$ and $\tilde{S}_i$, the iteration corresponding to S_i, $i = 1, 2$ from the previous section.

We investigate the smoothing property of $\tilde{S}_1$, $\tilde{S}_2$. It is easy to see, that

$$\tilde{R}_1 = r_1 \begin{bmatrix} 0\,0\,1 \\ 0\,0\,0 \\ 0\,0\,0 \\ 0\,0\,0 \\ 1\,0\,0 \end{bmatrix}, \qquad \tilde{R}_2 = r_2 \begin{bmatrix} 0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,0 \\ 1\,0\,0\,0\,0 \end{bmatrix}, \tag{12}$$

with suitable functions r_1 and r_2 depending on ϵ. Identifying $\tilde{R}_i$ with its eigenvalues, we get

$$\tilde{R}_1 = r_1(4\cos(k_1)\cos^2(k_2) - 2\cos(k_1) - 4\cos(k_2)\sin(k_1)\sin(k_2)),$$
$$\tilde{R}_2 = r_2(4\cos^2(k_1)\cos(k_2) - 2\cos(k_2) - 4\cos(k_1)\sin(k_1)\sin(k_2)).$$

Some properties of the ILU decompositions are given in

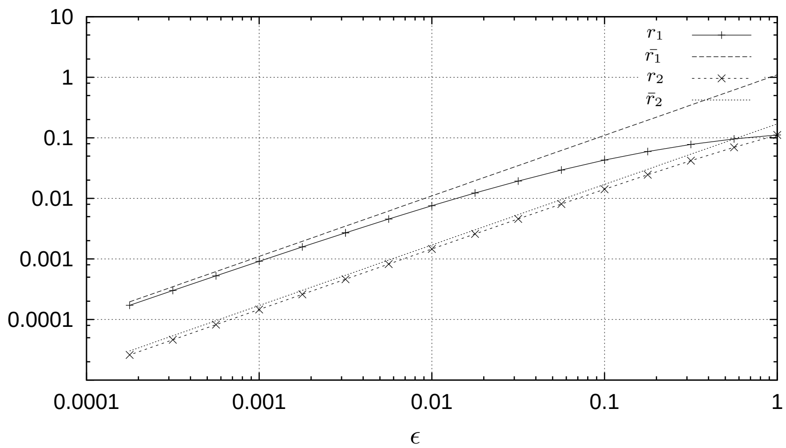

Fig. 1. Numerical evaluation of function r_1 and r_2 from (12) and their upper bounds $\bar{r}_1$ and $\bar{r}_2$.

Lemma 1 *The ILU decomposition* (11) *of* $\tilde{S}_1$ *and* $\tilde{S}_2$ *is well-defined and the following estimates hold*

$$\begin{aligned} 0 < r_1(\epsilon) \leq \bar{r}_1(\epsilon) &:= 1.10\epsilon, \quad \forall \epsilon \in]0,1], \\ 0 < r_2(\epsilon) \leq \bar{r}_2(\epsilon) &:= 0.17\epsilon, \quad \forall \epsilon \in]0,1]. \end{aligned}$$

Proof The proof cannot be given analytically. The convergence (11) has been verified numerically using double precision floating point arithmetics. The limiting values have been determined for 16 significant digits. The verification has been carried out for

$$\epsilon \in \{\xi \in]10^{-4}, 1] \,|\, \xi = (0.1)^{n/4},\, n \in \mathbb{N}_0\}.$$

The numerical evaluation of r_1 and r_2 are displayed in Fig. 1 together with their upper bounds $\bar{r}_1$ and $\bar{r}_2$.

Before we investigate the smoothing property of the iterations, we prove two technical lemma.

Lemma 2 *Let* $a, b, c, d \in \mathbb{R}$, $c \geq 0$, $d \in [0,1]$ *and* $f(x) := a + bx - c\sqrt{1 - d^2x^2}$. *Then it holds*

$$\min_{x \in [-1,1]} f(x) = \begin{cases} a - \sqrt{b^2/d^2 + c^2} & \text{if } b^2(1-d^2) < c^2 d^4 \\ a - |b| - c\sqrt{1-d^2} & \text{otherwise} \end{cases}.$$

Proof (i) Let $c > 0$ and $d = 1$. It holds $f'' > 0$ and therefore f is minimal for $x \in]-1,1[$ with $f'(x) = 0$, if existing, otherwise at the boundary of the interval. It holds

$$f'(x_0) = 0 \Leftrightarrow x_0 = -b/c\sqrt{1-x_0^2} \Leftrightarrow x_0 = -b/\sqrt{b^2+c^2} \in]-1,1[.$$

Therefore $\min_{x\in[-1,1]} f(x) = f(x_0) = a - \sqrt{b^2+c^2}$ hold in accordance with the assertion. (ii) Let $c > 0$ and $d \in]0,1]$. Then it holds $f(y/d) = a + by/d - c\sqrt{1-y^2}$, $y \in [-d,d]$. If

$$y_0 := -b/\sqrt{b^2+c^2d^2} \in]-d,d[\Leftrightarrow b^2(1-d^2) < c2d^4, \tag{13}$$

then it follows from (i) $\min_{y\in[-d,d]} f(y/d) = a - \sqrt{b^2/d^2+c^2}$ in accordance with the assertion. If (13) does not hold, the minimum is taken at the boundary of the interval $\min_{x\in[-1,1]} f(x) = a - |b| - c\sqrt{1-d^2}$. (iii) The cases $c \geq 0$ and $d = 0$ can be verified easily.

Lemma 3 *Let $i = 1,2$, $\epsilon \in [0,1]$, $0 \leq r_i\delta \leq 0.72\epsilon$ and $k_1, k_2 \in]-\pi,\pi]$. Then it holds*

$$A + \delta\tilde{R}_i \geq 0. \tag{14}$$

Proof First, we prove the assertion for R_2. It holds

$$\begin{aligned} A + \delta R \geq\ & \epsilon(2-\cos(k_1)) + (2-\cos(k_2)) + 4\delta r\cos^2(k_1)\cos(k_2) \\ & - 2\delta r(\cos(k_2) + 2|\cos(k_1)\sin(k_1)\sin(k_2)|). \end{aligned} \tag{15}$$

Using $x := \cos(k_1)$, $y := \cos(k_2)$, the right hand side of (15) reads

$$f := \epsilon(2-2x) + (2-2y) + \delta r(4x^2y - 2y - 4|x|\sqrt{1-x^2}\sqrt{1-y^2})$$

The assertion is equivalent to

$$\min_{r\in[0,c]} \min_{x\in[-1,1]} \min_{y\in[-1,1]} \min_{\epsilon\in[r/c,1]} f \geq 0, \quad \forall c,\delta > 0 \text{ with } c\delta \leq 0.72.$$

It holds

$$\begin{aligned} f_\epsilon :&= \min_{\epsilon\in[r/c,1]} f \\ &= 2r/c(1-x) + 2(1-y) + 2\delta r(2x^2y - y - 2|x|\sqrt{1-x^2}\sqrt{1-y^2}). \end{aligned}$$

Let $\tilde{a}_1 := 2 - 2xr/c + 2r/c$, $\tilde{b}_1 := -2 + 4\delta rx^2 - 2\delta r$ and $\tilde{c}_1 := 4\delta t|x|\sqrt{1-x^2}$, then $f_\epsilon = \tilde{a}_1 + \tilde{b}_1 y - \tilde{c}_1\sqrt{1-y^2}$ and due to Lemma 2

$$\begin{aligned} f_{y,\epsilon} :&= \min_{y\in[-1,1]} f_\epsilon = \tilde{a}_1 - \sqrt{\tilde{b}_1^2 + \tilde{c}_1^2} \\ &= 2(1 - xr/c + r/c) - 2(1+\delta r)\sqrt{1 - 4\delta rx^2/(1+\delta r)^2}. \end{aligned}$$

Let $\tilde{a}_2 := 2(1+r/c)$, $\tilde{b}_2 := -2r/c$, $\tilde{c}_2 := 2(1+\delta r)$ and $\tilde{d}_2^2 := 4\delta r/(1+\delta r)^2$, then $f_{y,\epsilon} = \tilde{a}_2+\tilde{b}_2 y-\tilde{c}_2\sqrt{1-\tilde{d}_2^2 x^2}$. Using Lemma 2 to determine $f_{x,y,\epsilon} := \min_{x\in[-1,1]} f_{y,\epsilon}$, one has to distinguish four cases. It holds

$$\tilde{b}_2^2(1-\tilde{d}_2^2) < \tilde{c}_2^2\tilde{d}_2^4 \;\Leftrightarrow 1/(\delta c)-4 < r/c < 1/(\delta c)+4. \qquad (16)$$

Case 1: Condition (16) is not fulfilled if $r/c \geq 1/(\delta c)+4$. This does not happen, since $\delta c > 0$ and $r/c \leq 1$.
Case 2: Condition (16) is not fulfilled if $\delta c \leq 1/4$ and $r/c \leq 1/(\delta c)-4$. Due to Lemma 2 we get

$$\begin{aligned} f_{x,y,\epsilon} &= 2(1+r/c)-2r/c-2(1+\delta r)\sqrt{1-4\delta r/(1+\delta r)^2} \\ &= 2-2|1-\delta r| \end{aligned}$$

and because of $\delta r = (\delta c)(r/c) \leq 1/4$ we get $f_{x,y,\epsilon} \geq 0$.
Case 3: Condition (16) and $\delta c < 1/5$ are never fulfilled, since $r/c \leq 1$.
Case 4: Condition (16) is fulfilled, if $\delta c \geq 1/5$ and $1/(\delta c)-4 < r/c < 1$. From Lemma 2 follows

$$f_{x,y,\epsilon} = 2(1+r/c) - 2(1+(\delta r)(r/c)\sqrt{1+4/(\delta c)(r/c)}).$$

Let $k := \delta c$, $z := r/c$ and $g(z) := z^2 + (4k-2/k)z + (8+1/k^2-8/k)$. Then $f_{x,y,\epsilon} \geq 0 \;\Leftrightarrow\; g(z) \leq 0$. Because of $g''(z) > 0$, $g(z) \leq 0$, $\forall z \in [0,1]$ is equivalent to $g(0) \leq 0, \wedge g(1) \leq 0$. (a) It holds $g(0) \leq 0 \;\Leftrightarrow\; k^2-k+1/8 \leq 0$, which is fulfilled for $k \in [0.2, 0.72]$. (b) It holds $g(1) \leq 0 \;\Leftrightarrow\; k^3+9/4k^2-5/2k+1/4 \leq 0$. It is easy to see, that the latter condition is fulfilled for $k \in [0.2, 0.72]$. Therefore $f_{x,y,\epsilon} \geq 0$ $\forall z = r/c \in [0,1]$ and $k = \delta c \in [0.2, 0.72]$. The proof of (14) follows the same line.

Now we can prove the robust smoothing property of $\tilde{S}_2$.

Theorem 1 *Let* $\epsilon \in]0,1]$ *and* A *defined by* (5)*. Then the ILU iteration* $\tilde{S}_2$ *fulfills the robust smoothing property* (3)*.*

Proof The first condition of (3) follows from (5), (12) and Lemma 1. From Lemma 1 we get $0 \leq r_2 \leq 0.18\epsilon$. Therefore Lemma 3 implies $A + 4\tilde{R}_2 \geq 0$. Hence $\tilde{W}_2 \leq 0.75A$.

Remark 1 The robust smoothing property for $\tilde{S}_1$ cannot be shown along the line of Theorem 1. The rest matrix $\tilde{R}_1$ decreases slower than $\tilde{R}_2$ with decreasing ϵ. From the estimate given in Lemma 1, it can be concluded only, that $A + \gamma_0\tilde{R}_1 \geq 0$ with $\gamma_0 \approx 0.7$, which is not sufficient. The numerical experiments given in the next section indicate, that the robust smoothing property does not hold for $\tilde{S}_1$.

5 Numerical results

In this section, we investigate the robustness of two different multigrid methods applied to three different model problems. The multigrid methods differ by the smoothers used, the model problems by the boundary condition and by the discretization.

We begin with the formulation of the model problems. Let $\Omega =]0,1[^2$ and $\epsilon \in]0,1]$. For all model problems a conforming Galerkin FEM is used. The mathematical model (1) of the first two model problems has been introduced in section 2. The two model problems differ by the discretization. To obtain the first one, equation (1) is discretized on the grid hierarchy (4) by means of $\mathcal{P}_1$ elements. The model problem is denoted by MP_1. The second model problem results from (1) using $\mathcal{Q}_1$ elements and the grid hierarchy

$$\mathbb{T}_0^s := \{\mathrm{square}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)\}, \tag{17a}$$
$$\mathbb{T}_l^s := \text{the uniform refinement of } \mathbb{T}_{l-1}^s,\ l > 0. \tag{17b}$$

For the inner DOF of the grid, it leads to the 9-point stencil

$$A = \frac{\epsilon}{6}\begin{bmatrix} -1 & 2 & -1 \\ -4 & 8 & -4 \\ -1 & 2 & -1 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} -1 & -4 & -1 \\ 2 & 8 & 2 \\ -1 & -4 & -1 \end{bmatrix}, \tag{18}$$

It is denoted by MP_2. To specify the third model problem, we define

$$\dot{\mathcal{L}}_2(\Omega) := \{u \in \mathcal{L}_2(\Omega) | \int_\Omega u dx = 0\},$$
$$\dot{\mathcal{H}}^1(\Omega) := \{u \in \mathcal{H}^1(\Omega) | \int_\Omega u dx = 0\}.$$

Let $u \in \dot{\mathcal{H}}^1(\Omega)$, $f \in \dot{\mathcal{L}}_2(\Omega)$ and

$$-\epsilon \partial_{xx} u - \partial_{yy} u = f, \quad \mathbf{x} \in \Omega, \tag{19a}$$
$$\partial_n u = 0, \quad \mathbf{x} \in \partial\Omega, \tag{19b}$$

where ∂_n denotes the partial derivative in direction of the outer normal. The solution is fully regular, i.e. $u \in \mathcal{H}^2(\Omega) \cap \dot{\mathcal{H}}^1(\Omega)$. Model problem MP_3 results from (19) discretized by $\mathcal{Q}_1$ elements on the grid hierarchy (17).

We investigate two V(1,1)-cycle multigrid schemes applied to model problems MP_1-MP_3. In all cases the canonical grid transfer operations are used and the exact solution is determined on grid level 1. The solution methods differ by the smoothers only. We investigate the multigrid schemes using the ILU iterations S_1 and S_2, denoted

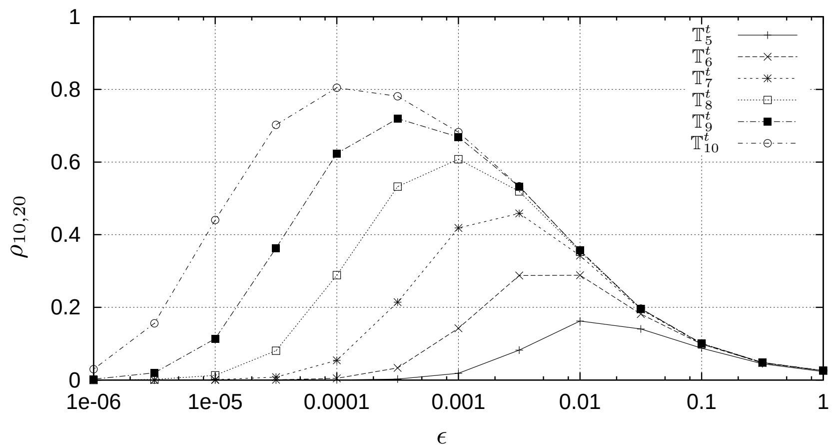

Fig. 2. Convergence rates of multigrid scheme 1 applied to MP_1 in dependence on ϵ.

by *scheme* 1 and *scheme* 2, resp. We consider the model problems on the grid level 5 to 10, corresponding to 32^2 to 1024^2 elements. We evaluate the multigrid convergence for

$$\epsilon \in \{\xi \in]10^{-6}, 1] \,|\, \xi = 10^{n/2},\, n \in \mathbb{Z}\}. \tag{20}$$

It is measured by

$$\rho_{10,20} := \left(\frac{\|d_{20}\|_2}{\|d_{10}\|_2} \right)^{1/10}, \tag{21}$$

where d_i denotes the defect after the i-th iteration. The discrete system of equations of MP_3 are singular. In this case adequate precautions have been taken to avoid numerical difficulties.

5.1 Numerical results for MP_1

We investigate the multigrid schemes for MP_1. The discretization is equivalent to a finite difference discretization. The convergence rates (21) of solution scheme 1 in dependence on ϵ on the different grid level are displayed in Fig. 2. They deteriorate with increasing grid level and do not seem to be bounded away uniformly from 1. Neglecting boundary conditions, the case corresponds to the usage of the ILU iteration $\tilde{S}_1$ from the previous section. The robust smoothing property could not be shown in that case, see Remark 1. The results are similar to the ones resulting from the ILU iteration S_0, which has

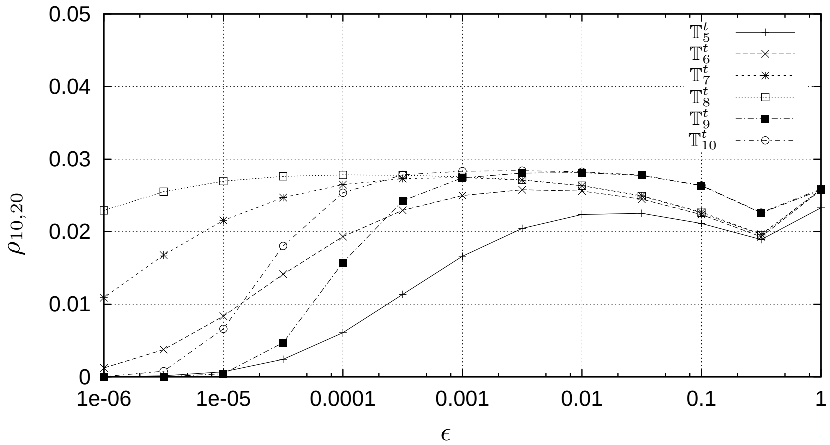

Fig. 3. Convergence rates of multigrid scheme 2 applied to MP_1 in dependence on ϵ.

been analyzed in [13]. Fig 3 shows the results for the multigrid scheme 2. The results indicate a very fast convergence of about $\rho \lesssim 0.03$ uniformly in ϵ and the grid level. Neglecting boundary effects, the ILU iteration used coincides with $\tilde{S}_2$ analyzed in section 4, which has been shown to fulfill the robust smoothing property (Theorem 1).

Remark 2 A modification of the ILU iteration S_0 has been discussed in e.g. [11,13]. For an appropriate modification, which has to be chosen experimentally, the usage of the iteration led to a robust multigrid convergence of about $\rho \lesssim 0.3$ [13]. The new method proposed here is superior to the modified ILU iteration.

5.2 Numerical results for MP_2

We investigate the multigrid schemes S_1 and S_2 for model problem MP_2. The discretization leads to the constant nine point stencil (18). The results for multigrid scheme 1 are displayed in Fig 4. As for MP_1, a uniform multigrid convergence is not obtained. In contrast to the previous case, the multigrid iteration diverges for sufficiently small ϵ and sufficiently fine grids. This is due to the fact, that for certain ϵ the stiffness matrix is no longer an M-Matrix and the ILU iteration divergences for sufficiently fine grids. The results for scheme 2 are similar to the ones obtained for MP_1. The multigrid convergence rate is uniformly bounded in ϵ and the grid level by $\rho \lesssim 0.03$.

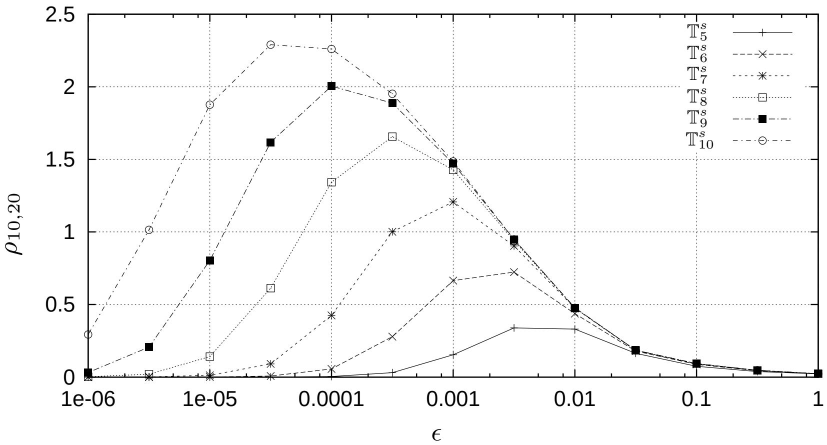

Fig. 4. Convergence rates of multigrid scheme 1 applied to MP_2 in dependence on ϵ.

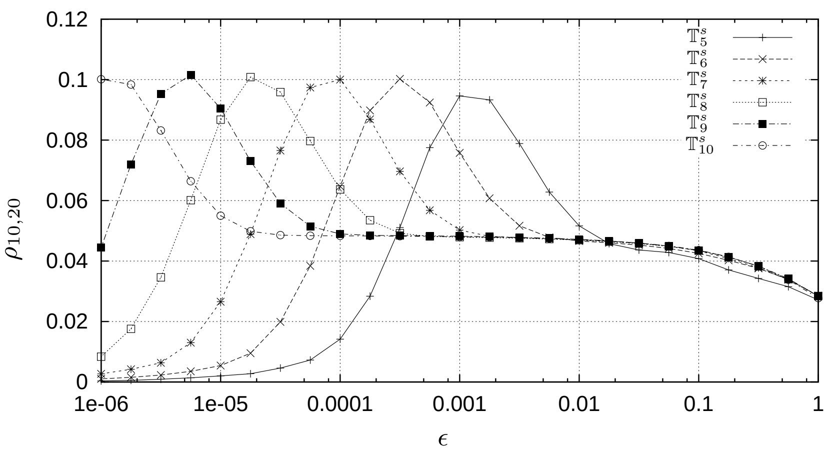

Fig. 5. Convergence rates of multigrid scheme 2 applied to MP_3 in dependence on ϵ.

5.3 Numerical results for MP_3

We investigate the multigrid schemes S_1 and S_2. The discretization leads, for interior DOF, to the same discretization stencil as for MP_2. Similar to MP_2, the multigrid scheme 1 diverges for certain ϵ and sufficiently fine grids. The convergence rates show a slightly more complex behavior than in the previous cases. Therefore, we chose a finer resolution for ϵ than (20). The convergence rates are given in Fig. 5. The convergence is uniformly bounded by $\rho \lesssim 0.1$.

6 Conclusions

In this paper we investigated 9-point variants of the ILU iteration as smoother in multigrid. For the 5-point discretization of the anisotropic Laplace operator a new proof is given relating the size of the restmatrix to the smoothing property. For an appropriate ordering excellent multigrid convergence results without modification of the ILU. The method carries over to the 9-point finite element discretization obtained on a tensor product grid.

Since no modification of the ILU iteration is necessary, an extension of the method to equations arising from systems of PDEs using a point block ILU method is easily possible. Applications to more complex model problems seem promising [3].

References

1. A. Brandt. Guide to multigrid development. *In: Lecture Notes in Mathematics, W. Hackbusch, U. Trottenberg (eds)*, 960:220 – 312, 1981.
2. W. Hackbusch. *Multi-Grid Methods and Applications*. Springer-Verlag, Berlin, Heidelberg, 1985.
3. K. Johannsen. *Numerische Aspekte dichtegetriebener Stroemung in poroesen Medien*. Habilitationsschrift, Heidelberg University, 2004.
4. R. Kettler. Analysis and comparison of relaxation schemes in robust multigrid and preconditioned conjugate gradient methods. *W. Hackbusch and U. Trottenberg, Eds., Lecture Notes in Mathematics*, 960:502 – 534, 1981.
5. N. Neuss. V-cycle convergence with unsymmetric smoothers and application to an anisotropic model problem. *SIAM. J. Numer. Anal.*, 35(3):1201–1212, 1998.
6. K.P. Oertel. Praktische und theoretische Untersuchungenzur ILU-Glättung bei Mehrgitterverfahren. Master's thesis, Bonn University, Germany, 1998.
7. K.P. Oertel and K. Stüben. Multigrid with of ilu-smoothing: systematic tests and improvements. *Proceedings of the Fourth GAMM-Seminar, Kiel*, pages 188–199, 1988.
8. R. Stevenson. Robust Multi-grid with 7-point ILU smoothing. *(to be published)*, 1992.
9. R. Stevenson. New estimates of the contraction number of V-cycle multi-grid with application to anisotropic equations. *In: Incomplete Decompositions, Algorithms, Theory and Applications, W. Hackbusch, G. Wittum (eds), Vieweg, Braunschweig*, pages 159 – 167, 1993.
10. R. Stevenson. Robustness of multi-grid applied to anisotropic equations on convex domains and on domains with re-entrant corners. *Numer. Math.*, 66:273–398, 1993.
11. R. Stevenson. Modified ilu as a smoother. *Numer. Math.*, 68:295–309, 1994.
12. P. Wesseling. A robust and efficient multigrid method. *W. Hackbusch and U. Trottenberg, Eds., Lecture Notes in Mathematics*, 960:614 – 630, 1981.
13. G. Wittum. On the robustness of ILU smoothing. *SIAM J. Sci. Stat. Comput.*, 10:699 – 717, 1989.